Finding the Earth's Radius with a Bullet

Andrew York
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In ancient times, a few individuals were able to calculate the radius of the earth in clever ways. Two-thousand years ago, Eratosthenes compared the sun's angle on the same calendar date a year apart in Syrene and then Alexandria, comparing the angles of the sun's rays, which enabled him to calculate an approximation of the polar circumference of the earth, with an error rate between 1 and 20 percent (depending on the actual length of the measuring unit at the time, stadia). In the 10th century, Al-Biruni measured the earth's radius by measuring the elevation angle of a mountain from two points in a line. With the help of an astrolabe and some trigonometry, he was able to calculate the earth's radius, also to a disputed degree of accuracy.

Today, knowing the mass of the earth, we can easily find the radius of earth by using Newton's gravitational equations:

$$r = \frac{1}{\sqrt{\frac{g}{GM}}}$$

where r = radius of earth, g = earth's gravitational acceleration, G = gravitational constant, and M = mass of earth

But I thought it would be interesting to find the radius of a planet of unknown mass and size, using only the information produced by a projectile. As a thought experiment, imagine you are on this unfamiliar planet, and you don't know its radius, but would like to find it. Imagine further you have only one device to help you, a machine that fires a projectile at any speed you like, parallel to the ground.

We continue firing projectiles with increasing velocity over a flat area of the planet's surface, and measuring the amount it drops due to gravity after one second. We increase the velocity until we find that the projectile stays the same distance from the ground after one second, meaning that the amount it has fallen due to the planet's gravitational acceleration equals the amount the planet's surface has curved away from the projectile's starting point. This would be the surface orbital velocity for this planet. We measure and make a note of the distance the projectile traveled parallel to the surface in one second, and we will call this distance b (represented in figure 1 by the red arclength hj).

Next we need to drop some things and measure g, the acceleration of the planet. Since the distance an object falls is $\frac{1}{2}gt^2$, we know that the object will fall half the distance of g

in one second. We'll use the letter a for this distance of $\frac{1}{2}g$ (represented in figure 1 by the red line jk).

So now we have two values; b, the distance traveled by the projectile at orbital velocity in one second (red arclength hj), and a, the distance the projectile has dropped due to gravity along its path (line jk). Using only these two values, we will find the radius of the planet.

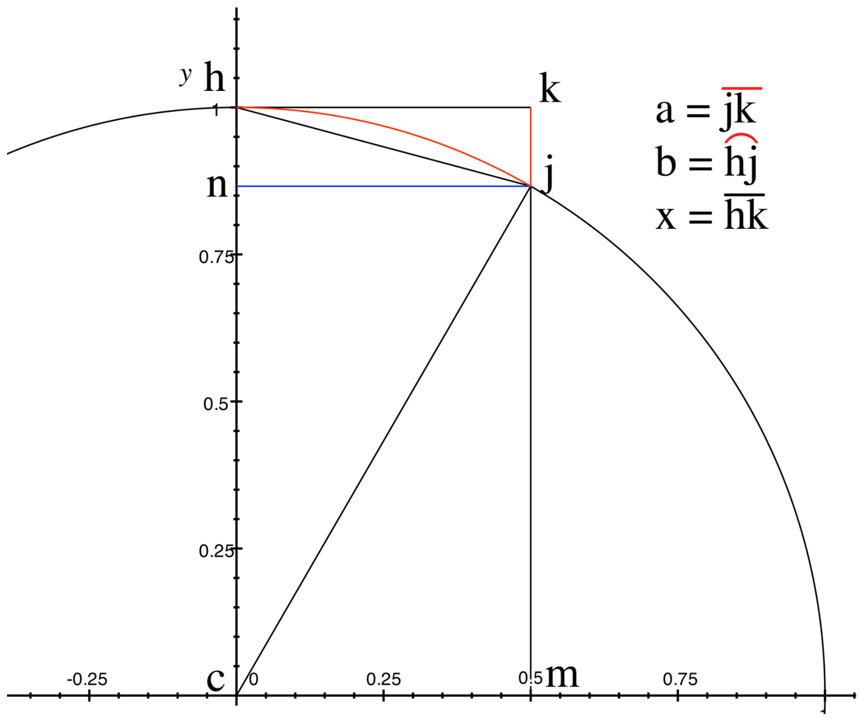

figure 1

If we were using a circle with a radius of 1, we would find that the values of a and b would be quite simple to work with:

$$a = 1 - \sqrt{1 - x^2} \quad \text{and} \quad b = \arcsin(x)$$

The value for b would be obvious, since any length around the surface of a circle is equivalent to its angle in radians, when the radius of the circle is 1. For finding a (or jk) we can do this in two ways when the circle has a radius of 1. We can subtract cos(arcsin(x)), which is the same as cos(b), from the radius value of 1, and since $x = hk = nj$, taking cos(arcsin(nj)) gives us the length of line nc, and line jm is equivalent to line nc. Then subtracting the length of line jm from 1, the circle's radius, we get the value for a, or line jk.

We can also use the Pythagorean theorem to find the adjacent side of $\triangle ncj$. Again, since $x = nj$, and nj is the opposite side of $\triangle ncj$, and since the hypotenuse cj is equal to 1, we take the square root of $1 - x^2$. This gives us the length of adjacent side nc, which also equals jm. We now take $1 - jm$, which is the same as $1 - \sqrt{1 - x^2}$, and we again have the length of a, or jk.

However, we are on our strange planet, and the radius is certainly not equal to 1. The only data we have is the value of b (the distance the projectile travels at orbital velocity in one second), and the value of a (the amount it falls due to gravitational acceleration in one second). We cannot use arcsin to calculate the angle that the value of b represents, since that distance is quite large and the represented angle would be dependent on the circumference of the planet. So we have no indication of the unique angle from the planetary center represented by the arc of the projectile's path.

The ratio a/b is associated with only one unique angle between zero and $\pi/2$. For this unique angle, the size of the planet will determine the actual length of b, and also of a (the distance the projectile has fallen in one second). As mentioned above, the unique angle with which the ratio a/b is associated can be found most easily when the radius of the planet is equal to 1. Then the circumference of the planet would be equal to 2π, and so the length of b would be the length in radians around the circumference of the planet, which would also be its angle in relation to the center of the planet.

For example, in figure 1, the value of b is clearly 0.52359, or 30°, since the angle involved is $\arcsin(x)$, or $\arcsin(0.5)$. The length of a can be found by $1 - \cos(b)$, which would be $1 - 0.866024 = 0.13397$. The ratio a/b would then be 0.13397 / 0.52359, or $a/b = 0.25587$.

A problem arises when we try to find the angle represented by the distance the projectile travels in one second. To illustrate, if the planet has a circumference C of 1,000,000 meters then its radius r is $C/2\pi$, or 159155 meters. Then the value of b that would represent an angle of 0.52359 radians would be $0.52359 * r$, or 83333.33 meters, and the corresponding value of a would be $0.13397 * r$, or 21322.72 meters.

It's important to remember that a/b is a ratio, and whether a/b is 21322.72 / 83333.33, or 0.13397 / 0.52359, the ratio is the same, and both sets of values will return the same number here, or 0.25587, which is intimately related to a unique angle that corresponds only to that value of a/b. It is clear that any size planet will return the same value for a/b with a fixed angle $\angle hcj$.

So, just knowing that the projectile traveled 83333.33 meters and fell 21322.72 meters does not yet help us find the angle needed to calculate the planet's radius. We need a formula based on the ratio a/b that we can use to find the unique angle to which it points. Here is a formula that will take care of this problem:

$$\frac{a}{b} = \frac{1 - \cos(\arcsin(x))}{\arcsin(x)}$$

As we see in the graph in figure 2, there is only one unique value of x that will return the ratio a/b, and this value of x is also the length of lines hk, nj and cm in figure 1, when the radius of the circle is equal to 1. Taking $\arcsin(x)$ will give us the angle on the planet's surface represented by b, the distance the projectile traveled in one second at orbital velocity. Finding the unique angle for b, which is $\arcsin(x)$, is equivalent to reducing the size of the planet to a radius of 1.

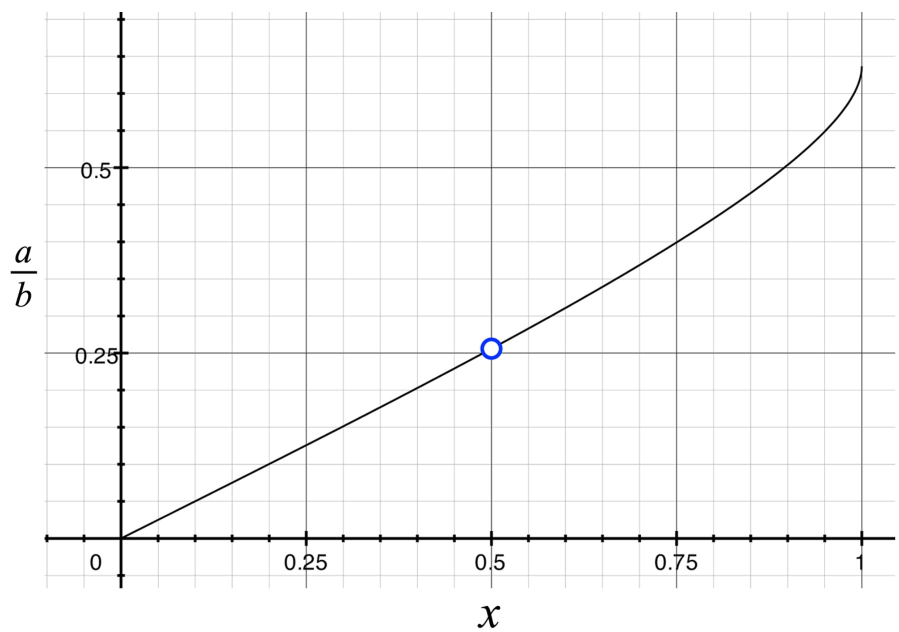

figure 2

But instead of graphing all values of $0 < x \geq 1$ to find the true angle, and solve for x this way, I liked the challenge of making an approximation equation for finding the value of x directly from the ratio a/b.

$$\left(\left(\frac{2a}{b}\right) - \left(\left(\sqrt[\pi]{\frac{\pi}{2}}\right)\frac{a}{b}\right)^{\left(\pi + \left(2 - \frac{\pi}{19}\frac{a}{b}\right)\frac{a}{b}\right)}\right) \approx x$$

figure 3

This equation will return an approximation of the value of x, or line hk in figure 1, when the ratio a/b is plugged into it. Since this line is equivalent to the line nk, which is the opposite side of $\triangle ncj$, we can now take arcsin(x) to find the unique angle represented by arclength b.

As an example, let's look at how using the approximation equation in figure 2 would play out on the earth. On earth, a projectile in orbit at ground level would need to be traveling at 7909 m/s to be in a ground level orbit, and so on earth $b = 7909$ m. And the gravitational acceleration on earth is 9.80665 m/s^2, so using the formula $\frac{1}{2}gt^2$ to calculate the distance the projectile would fall after one second, gives us $a = 4.903325$ m. Putting a/b, or 4.903325 / 7909, into the equation in figure 3, we get 0.00123994, which represents the length of x (or line hk) for a circle with a radius of 1. Taking the arcsin of 0.00123994, we arrive at the same number - because the arclength on earth of 7909 meters is such a small length along the circumference of the earth, it is almost a straight line. And since the angle represented by a (4.9 meters) to the starting point of the projectile is also such a tiny angle in relation to the curve of the earth, both line hk and line hj are almost the same value as b, or arclength hj. What this means is that the angle represented by the value of b along the earth's surface, as radiating from the earth's center, is also 0.00123994.

Now that we have found the angle that 7909 meters represents around the circumference of the earth, we can now find the radius of the earth. First we divide 2π / 0.00123994, giving us 5067.33. That is how many segments of 7909 meters there are around the earth's circumference. Multiplying 5067.33 * 7909 gives 40077514, a very close approximation of the circumference of the earth at the equator. Dividing this by 2π to get the radius returns 6378534 meters, a very good approximation of the radius of the earth.

The entire equation to find the radius r of earth, using only the data gathered from the projectile in the form of the ratio a/b is:

$$\frac{\left(2\pi \div \arcsin\left(\left(\frac{2a}{b}\right) - \left(\left(\sqrt[\pi]{\frac{\pi}{2}}\right)\frac{a}{b}\right)^{\left(\pi + \left(2 - \frac{\pi}{19}\frac{a}{b}\right)\frac{a}{b}\right)}\right)\right)b}{2\pi} \approx r$$

One might notice that with very small angles represented by the projectile's traveling distance, as they are on earth, we don't need to do such a laborious equation. We can simply take $2(a/b)$ and receive almost the same result as the equation in figure 3. But if the distance traveled by the projectile becomes a greater percentage of the planet's circumference, the error of using $2(a/b)$ becomes greater and greater. Look at figure 1 which shows an angle of 0.52359 radians, giving us an a/b value of 0.25587. Then $2(a/b)$ returns 0.5117, which is noticeably less than the correct value of 0.52359, and the error will increase as the angle b along the circle increases. But, using the equation in figure 3, plugging in an a/b value of 0.25587, we would generate an x value of 0.499966, the arcsin of which is 0.52356. Not the exact value of arcsin 0.5, but very close for an approximation, with an error of only 0.000034.

And with this we can arrive at the radius of the earth (or any other planet we might find ourselves on) just by firing a bullet, as long as that bullet goes at orbital velocity. Not an eventuality that is likely to confront us, but it was enjoyable to work it out nonetheless.

www.andrewyork.net/Math/mathmusings.php

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