A New Identity for the Natural Logarithm
for all values of x, greater than 1 and less than 1

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In my previous article "Throwing a Curve at Logarithmic Spirals" I discussed a new way to generate logarithmic or equiangular spirals without using e, the base of the natural logarithm. In place of the standard formula $r = e^\theta$ for generating logarithmic spirals, the effective sigma summation I discovered to generate the angle θ in polar coordinates to describe the growth of a spiral with a constant equiangle α at $\pi/4$ is

$$\theta = \sum_{n=1}^{u} \frac{1}{n} - \sum_{p=1}^{ru} \frac{1}{p}$$

where r=radius, and u represents the number of summation iterations, giving the level of resolution required.

To reduce the formula to a single summation, I created separate values for variables a and b that allow my equation to work with r values $0 < r < 1$, and also $r > 1$. This ability to work with values less than 1 as well as greater than 1 will prove significant.

$$\theta = -\frac{|ru - u|}{ru - u} \sum_{n=a}^{b} \frac{1}{n}$$

where,

$$a = \frac{(ru + u) - (|ru - u|)}{2}$$

$$b = a + (|ru - u|)$$

In a polar graph, this will plot the spiral correctly for values of r both $r < 1$ and $r > 1$. In addition, we also need the switch $\frac{|ru - u|}{ru - u}$ to reverse the polarity of angle θ for values $r < 1$, else that segment of the spiral will be reversed from proper orientation. The negative sign at the beginning is simply to have the spiral curve in a clockwise direction, consistent with my previous article. It should be noted that the value of u, while still an indicator of the resolution of the output, does not directly represent the number of sums that will be iterated with this single sigma equation.

Identity for the natural logarithm

The standard formula for a logarithmic spiral at an equiangle of π/4 is $r = e\hat{}\theta$, and the inverse of this formula using $f(r)$ instead of $f(\theta)$ is then $\theta = ln(r)$, which also creates a logarithmic spiral with an equiangle of π/4. Therefore, it stood to reason that $\theta = \sum_{n=1}^{u} \frac{1}{n} - \sum_{p=1}^{ru} \frac{1}{p}$ might be a new identity for $ln(r)$, since both $\theta = \sum_{n=1}^{u} \frac{1}{n} - \sum_{p=1}^{ru} \frac{1}{p}$ and $ln(r)$ generated the same numbers even at moderate resolution values of u.

Now, the Taylor series is commonly invoked as an identity for finding $ln(x)$. A common version is

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x - 1)^n$$

However, this identity is only accurate if the value of x is between 0 and 1. It becomes extremely inaccurate above values of 1, rendering it useless for values $x > 2$.

I found that my equation is indeed an identity for $ln(x)$, and approximates the value of the natural logarithm not only for values $0 < x < 1$, but also for $x > 1$. Using my single summation equation above, I found some adjustments to increase the accuracy.

$$\ln(x) = -\frac{|xu - u|}{xu - u}\sum_{n=a}^{b} \frac{1}{n+\frac{1}{2}}$$

where,

$$a = \frac{(xu + u) - (|xu - u|)}{2}$$

$$b = a + (|xu - u|) - 1$$

The range of the summation defined by a and b is determined by multiplying our start value a by the value of x when x>1, or the reciprocal of x when 0<x<1. For example, if we want to find $ln(x)$ where x=0.2, and our start value a = 1000, then a * (1/x) = b, so b would equal 1000 * (1/0.2), or 5000. (1 is then subtracted from b to increase the accuracy of the identity, so the iterations of n to be summed would then be from n=1000 to 4999.) The level of resolution of the approximation of $ln(x)$ is represented by u. The higher the value of u, the greater the resolution of $ln(x)$ will be.

In contrast to the Taylor series commonly used as an identity for finding $ln(x)$, my identity is fairly simple, with no exponents involved in calculating the sigma summation. (I'll call it the 'harmonic identity' from here on, due to its connection to the harmonic series, as I will explain below). There is also no limitation to the positive value of x for which we can find the natural logarithm. However, the Taylor series needs less iterations to return a more accurate value for `ln`, while the harmonic identity requires more iterations for comparable accuracy when finding values for $x < 1$. As an example, if the value of x is 0.2 and the number of sum iterations is $u = 150{,}000$, the harmonic identity returns a value of -1.6094379124, which is the value of `ln`(0.2) to an accuracy of one ten-billionth. The Taylor series also returns this value, but at only 110 iterations.

The situation is somewhat reversed when the harmonic identity finds `ln` values of $x > 1$ and the taylor finds the reciprocal equivalent `ln` values of $x < 1$. For example, at `ln`(0.001), the taylor series takes 10,000 iterations to return a value that is accurate to five decimal places, while the harmonic identity returns the same accuracy with 90 iterations for the reciprocal value `ln`(1000), which is the same as `ln`(0.001) except of opposite sign.

Moreover, at $x = 2$ and $u = 20{,}000$, the Taylor series returns a value for `ln`(2) of 0.69312218, which is accurate to four decimal places of the true value `ln`(2)=0.6931471805. At the same number of iterations the harmonic identity returns the correct value of 0.693147180, which is accurate to nine decimal places.

At higher values of x, the taylor series' results deteriorate dramatically. At $x = 5$, with only 10 iterations, the Taylor series returns a value for `ln`(5) of -8.2e+4, which is far from the true value of `ln`(5) = 1.6094379. At 10 iterations the harmonic identity returns 1.609, already accurate to three decimal places. At higher iterative values of u, the Taylor series results become undefined. In contrast, at $x = 5$ and $u = 10{,}000$, the harmonic identity returns a value for `ln`(5) of 1.609437912, which is correct to nine decimal places.

Independence of scale within the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

A very interesting observation is that the harmonic identity works in a scale-independent manner within the harmonic series. One can begin the calculation at any point in the harmonic series and still achieve an approximation of $ln(x)$. This almost holographic quality of the identity is remarkable.

The following approach simply and accurately generates the natural logarithm using scale-independent sections of the harmonic series. These sections will be summed at reciprocal values precisely in-between the whole integers. For example, in finding the natural logarithm of 0.5, with a u value of 1000, the harmonic identity would be summing from n=500 + ½ to 999 + ½. By summing using the equation

$$\sum_{n=xu}^{u-1} \frac{1}{n+\frac{1}{2}}$$

the section of the harmonic series to be summed would be the reciprocals of the series n = 500.5 + 501.5 + 502.5 . . . 999.5. By summing in this manner at values of n + ½ we achieve an accurate value for the natural logarithm for any value $x < 1$. If we change the u value to 600,000, then the summing range would be from 300,000 + ½ to 599,999 + ½. We immediately notice that the relationship is the same anywhere within the harmonic series. We can start at any point n + ½ in the harmonic series and sum to $2n$ - ½, and we will obtain the same result, in this case $\ln(0.5)$.

Similarly, for a value of $x = 0.2$, we can start at any value for n + ½ and sum until $5n$ - ½. This leads to the realization that we can find the natural logarithm of any number x where $0 < x < 1$ by:

$$-\sum_{n=u}^{\frac{1}{x}u-1} \frac{1}{n+\frac{1}{2}}$$

and for x values $x > 1$ by:

$$\sum_{n=u}^{xu-1} \frac{1}{n+\frac{1}{2}}$$

where u is any positive number >1, with larger values of u returning more accurate approximations of the natural logarithm.

Let's call the harmonic series A. We know that:

$$A = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$$

When we add 1/2 to each value of n, we then have a summed series from the reciprocals of 1.5, 2.5, 3.5, etc. Let's call this series *B:*

$$B = \sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{2}} = \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} \ldots$$

The amount that each element of series *B* is less than each corresponding element of the harmonic series *A* is:

$$\frac{1}{2n^2 + n}$$

So,

$$\frac{1}{n} \in A \text{ and } \frac{1}{n+\frac{1}{2}} \in B, \text{ then } \frac{1}{n} - \frac{1}{n+\frac{1}{2}} = \frac{1}{2n^2 + n}$$

For example, if we look at the fourth element of each series, we have

$$\frac{1}{4} - \frac{2}{9} = \frac{1}{2n^2 + n} = \frac{1}{2(4)^2 + 4} = \frac{1}{36},$$

which is the difference between the 4th element of series *A* and series *B*. This is the corresponding relationship of series B to the harmonic series (*A*).

It is also curious that if we sum the harmonic series to any value x, and then subtract γ, the Euler-Mascheroni constant of 0.57721566, we are left with an approximation of $ln(x)$.

$$ln(x) = \sum_{n=1}^{x} \frac{1}{n} - \gamma$$

The value of x has to be quite large for the approximation of $ln(x)$ to become reasonably close, and there is no flexibility in increasing the resolution at lower values. However, the relationship between γ and the natural logarithm is of great theoretical interest.

We also note that if we sum series *B* in the same manner to a value x, and then subtract the summed value from $ln(x)$, we get a number that seems to approach 0.03649:

$$ln(x) - \sum_{n=1}^{x} \frac{1}{n + \frac{1}{2}} \approx 0.03649$$

Identity for the base of the natural logarithm

It became obvious that we can use the harmonic identity to also create a very good approximation of e. By using the two separate sigma summations above, we can transform these identities into equations to generate e quite well with the following identities:

$$e = \sqrt[-\sum_{n=ux}^{u-1} \frac{1}{n+\frac{1}{2}}]{x}$$

for values of x where $0 < x < 1$, and

$$e = \sqrt[\sum_{n=u}^{ux-1} \frac{1}{n+\frac{1}{2}}]{x}$$

for values of x where $x > 1$, and u = resolution, or number of summation iterations.

Or, we can combine the two identities to solve any value of x, whether less than 1 or greater than 1, with the identity

$$e = \sqrt[\frac{|ux-u|}{ux-u} \sum_{n=a}^{b} \frac{1}{n+\frac{1}{2}}]{x}$$

where,

$$a = \frac{(xu + u) - (|xu - u|)}{2}$$

$$b = a + (|xu - u|) - 1$$

and u = resolution.

This may also be expressed as

$$e = x \wedge \frac{1}{\frac{|xu - u|}{xu - u} \sum_{n=a}^{b} \frac{1}{n + \frac{1}{2}}}$$

Conclusion

Thus we have uncovered the harmonic identity, a new identity based on the harmonic series, for approximating the natural logarithm for any positive value of x without limitation, and for approximating the value of e, the base of the natural logarithm.

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